circular flap to the shroud further increases the shroud's exit area A_e and, as expected, also increases r. Every shroud has its own optimal load factor C_D; in general, an increase in the shroud exit area (with or without flaps) reduces the optimal value of C_D (see Fig. 4).

Conclusions

The main object of this work was to arrive at a compact shroud configuration. The shrouds of models, C (i), C (ii), and C (iii), had "economical geometry," i.e., a fairly short length relative to the turbine diameter (total length to throat diameter ratio of the order of 3:1). Furthermore, when the wind direction changes, an increase in r can be expected in the shrouded aerogenerator as long as the yaw angles are within the shroud's stall range (about $\pm 25°$). Increasing the shroud exit area will increase the power ratio obtained under given stream conditions. The addition of a circular wing (flap) causes a significant improvement in the shroud performance (increase in r) but reduces significantly the stall range of the shroud. (Now r_{max} will be reached at $\theta = 0°$, see Fig. 3.) Three additional advantages of the use of shrouds for aerogenerators are 1) the axial velocity at the turbine section is higher than the freestream velocity, thereby making it possible to build smaller rotors that rotate at higher rpm; 2) by virtue of the enclosure produced by the shroud, tip losses can be significantly reduced; and 3) axial flow turbines are appropriate for use with shrouds. It has been shown that such turbines are capable of producing stable output at varying wind velocities without requiring a pitch control mechanism.[10]

References

[1] Glauert, H., *The Elements of Airfoil and Airscrew Theory*, Cambridge University Press, Cambridge, 1959.

[2] Golding, E. W., *Generation of Electricity by Wind Power*, E. and N. Spoon, Ltd., London 1955.

[3] Putnam, P. C., *Power from the Wind*, D. Van Nostrand, New York, 1948.

[4] "New Sources of Energy," Proceedings of the Conference, Rome, Aug. 21-31, 1961, U. N. Publications Sales No. 63.1.41, E/Conf. 3/8 7:188, 1964.

[5] Kogan, A. and Nissim, E., "Shrouded Aerogenerator Design Study, Two-Dimensional Shroud Performance," *Bulletin of the Research Council of Israel*, Vol. 11c, 1962, pp. 67-88.

[6] Kogan, A. and Seginer, A., "Shrouded Aerogenerator Design Study II, Axisymmetrical Shroud Performance," Proceedings of the Fifth Israel Annual Conference on Aviation and Astronautics, Israel 1963.

[7] Kogan, A. and Seginer, A., "Shrouded Aerogenerator, Design Study II, Axisymmetric Shroud performance," Dept. of Aeronautical Engineering, Technion, T.A.E. Rept. 32, 1963; also [7b] Kogan, A. and Seginer, A., "Final Report on Shroud Design," Dept. of Aeronautical Engineering, Technion, T.A.E. Rept. No. 32A, 1963.

[8] Igra, O., "Shrouds for Aerogenerators," AIAA paper 76-181, Washington, D. C., 1976.

[9] Fletcher, H. S., "Experimental Investigations of Lift, Drag and Pitching Moment of Five Annular Airfoils," NASA TN 4117, 1957.

[10] Igra, O., "Design and Performance of a Turbine Suitable for an Aerogenerator." *Energy Conversion*, Vol. 15, 1976.

Generalized Inverse of a Matrix: The Minimization Approach

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Introduction

THE generalized inverse of a matrix[1] has been used extensively in the areas of modern control, least-square estimation, and aircraft structural analysis. In a recent paper,[2] certain practical aspects of the generalized inverse were discussed. It is the purpose of this engineering note to extend the results of Ref. 2 by presenting a unified framework that provides geometric insight and highlights certain optimal features imbedded in the generalized inverse.

Consider the algebraic matrix equation

$$y = Ax \tag{1}$$

where A is an $n \times m$ constant matrix, y is a given n vector, and x is an m vector to be determined. For the trivial case where $n = m$ and A is a nonsingular matrix, i.e., rank $(A) = n$, a unique solution to Eq. (1) exists and is given by

$$x = A^{-1} y \tag{2}$$

where A^{-1} designates the inverse of A.

For the case $n \neq m$, the expression of x in terms of y involves the generalized inverse of A, denoted A^+, and thus

$$x = A^+ y \tag{3}$$

In the following cases it will be shown that A^+, for either $n > m$ or $n < m$, may be viewed as a solution to a certain minimization problem.

Case A: $n > m$

With no loss of generality it can be assumed that A is of full rank, i.e.,

$$\text{rank}(A) = m \tag{4}$$

If, however, rank $(A) < m$, it is possible to delete the dependent columns of A, set the respective unknown components of x equal to zero, and reduce the problem to the case in which Eq. (4) is satisfied. Since the m columns of A do not span the n dimensional space R_n, an exact solution to Eq. (1) cannot be obtained if y is not contained in the subspace spanned by the columns of A. Thus, one is motivated to seek approximate solutions, the best of which is the one that minimizes the Euclidian norm of the error. Let the error e be given by

$$e = y - Ax \tag{5}$$

Then let z be given by

$$z = \|e\|^2 = e^T e = (y - Ax)^T (y - Ax) \tag{6}$$

where the superscript T denotes the transpose. Evaluation of the gradient of z with respect to x yields

$$\partial z / \partial x = -2A^T y + 2A^T A x = 0 \tag{7}$$

while the Hessian matrix is

$$\partial^2 z / \partial x^2 = 2A^T A \tag{8}$$

From Eq. (7), x is given by

$$x = (A^T A)^{-1} A^T y \tag{9}$$

By virtue of A having full rank, $(A^T A)$ is a positive definite matrix. Thus $(A^T A)^{-1}$ exists and the Hessian matrix is positive definite, implying that a minimum was obtained. In this case $(n > m)$ the generalized inverse of A is given by

$$A^+ = (A^T A)^{-1} A^T \tag{10}$$

It is interesting to note that if y is contained in the subspace spanned by the columns of A, Eq. (9) yields an exact solution to Eq. (1), i.e., $\|e\| = 0$. The optimal feature of Eq. (9) has

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found extensive applications in data processing for least-square approximations.[3] In closing it should be noted that Eq. (9) can be obtained by invoking the Orthogonal Projection Lemma,[4] thus providing a geometric interpretation to the optimal features of Eq. (10).

Case B: $n<m$

Again, with no loss of generality, it can be assumed that A is of full rank, i.e.,

$$\text{rank}(A)=n \tag{11}$$

If, however, rank$(A)<n$, it implies that some of the equations are merely a linear combination of the others and therefore may be deleted without loss of information, thereby reducing the case rank$(A)<n$ to the case rank$(A)=n$. Moreover, if A is a square singular matrix, it can be reduced to case B after proper deletion of the dependent rows of A.

As posed, Eqs. (1) and (11) with $n<m$ yield an infinite number of solutions, the "optimal" of which is the one having the smallest norm. Therefore, one is confronted with a constrained minimization problem, where the minimization of $\|x\|$ (or equivalently $(1/2)\|x\|^2$) is to be accomplished subject to Eq. (1). Adjoining the constraint, via a vector of Lagrange Multipliers (λ), to the fundamental to be minimized we obtain

$$H=\tfrac{1}{2}\|x\|^2+\lambda^T(y-Ax)=\tfrac{1}{2}x^Tx+\lambda^T(y-Ax) \tag{12}$$

Evaluating the respective gradients

$$\partial H/\partial x=x-A^T\lambda=0 \tag{13}$$

$$\partial H/\partial \lambda=y-Ax=0 \tag{14}$$

From Eq. (13)

$$x=A^T\lambda \tag{15}$$

Substitution of Eq. (15) in Eq. (14) yields

$$y=AA^T\lambda \tag{16}$$

or

$$\lambda=(AA^T)^{-1}y \tag{17}$$

The existence of $(AA^T)^{-1}$ is guaranteed by virtue of Eq. (11). Substitution of Eq. (17) in Eq. (15) yields

$$x=A^T(AA^T)^{-1}y \tag{18}$$

In this case $(n<m)$ the generalized inverse of A is given by

$$A^+=A^T(AA^T)^{-1} \tag{19}$$

For the sake of completeness it should be noted that the norm minimization of $\|e\|$ and $\|x\|$ of Eqs. (6) and (12), respectively, can be performed relative to extended vector norms, where a norm of a vector w is defined as w^TQw and Q is a compatible positive definite weighting matrix. By so doing, one can choose the relative emphasis of the magnitudes of the vector components being minimized.

For case A, Eq. (6) becomes

$$z=e^TQe=(y-Ax)^TQ(y-Ax) \tag{20}$$

with the solution

$$x=(A^TQA)^{-1}A^TQy \tag{21}$$

For case B, Eq. (12) becomes

$$H=\tfrac{1}{2}x^TQx+\lambda^T(y-Ax) \tag{22}$$

with the solution

$$x=Q^{-1}A^T(AQ^{-1}A^T)^{-1} \tag{23}$$

A unified framework has been presented showing that the generalized inverse of a matrix can be viewed as a result of a minimization problem leading to a practical engineering interpretation.

References

[1]Penrose, R., "A Generalized Inverse for Matrices," *Proceedings of the Cambridge Philosophical Society*, Vol. 51, 1955, pp. 406-413.

[2]Hassig, H. J., "Practical Aspect of the Generalized Inverse of a Matrix," *AIAA Journal*, Vol. 13, Nov. 1975, pp. 1530-1531.

[3]Sorenson, H. W., "Kalman Filtering Techniques," *Advances in Control Systems*, edited by C. T. Leondes, 1966, pp. 219-340.

[4]Luenberger, D. G., *Optimization by Vector Space Methods*, Wiley, New York, 1969.

Resizing Procedure for Structures Under Combined Mechanical and Thermal Loading

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Introduction

PROBABLY the most widely used approach for sizing of flight structures under strength and minimum gage constraints is fully-stressed design (FSD). In this method the structural sizes are iterated with the step size depending on the ratio of the total stress to the allowable stress.[1-3] The FSD procedure traditionally is used to obtain, at a reasonable computational cost, designs which, if not at a minimum weight, are at least acceptably close to the minunum weight.[2]

Almost all of the experience with FSD has been with structures primarily under mechanical loading as opposed to thermal loading. The temptation in including thermal loads in FSD is simply to continue to use the total stresses in computing the iteration step size. This approach seems satisfactory when mechanical stresses dominate the thermal stresses.[4] Convergence may be slow, however, when thermal stresses are comparable to mechanical stresses. The slowed convergence is associated with relative insensitivity of the thermal stresses to changes in structural sizing. Therefore, procedures are needed which take into account the differing responses of thermal and mechanical stresses to changes in structural sizes.

An improved variant of FSD for uniaxial stress members was described in Ref. 4 and demonstrated for automated sizing of truss-type structures. For problems having sub-

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